ALTERNATIVE DEFINITIONS OF ELASTICITY OF SUBSTITUTION: REVIEW AND APPLICATION

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ABSTRACT

The concept of elasticity of substitution appeared in economic literature in the early 1930s. After that, the concept has suffered much criticism and alternative definitions were suggested. The critics and the various definitions do not seem to be well known given the small number of references we find in the literature with respect to this problem. The objective of this paper is to present a quick review of the concept of elasticity of substitution and to present its alternative definitions, which are generalizations of the two-factor case. It is emphasized that, when there are three or more factors of production, the concept of elasticity of substitution is not free of ambiguity. Besides the very well known *Allen* elasticity of substitution, alternative definitions of *Allen-Uzawa, Hicks, Morishima and McFadden* are presented and discussed. An attempt is made to show the differences among the definitions and their relevance as an instrument of analysis of factor substitution in the productive process. An application was done with data from electrified farms of the State of Minas Gerais. A *translog* production function was estimated with the objective of showing the nature and degree of substitution between electrical energy and other factors of production. The results showed great coherence among the various types of elasticities, but very important differences were found. *Morishima* elasticity provides more information

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about the substitution process of factors of production.

Key words: Marginal rate of substitution, elasticity of substitution, *Allen* elasticity of substitution, *Hicks* elasticity of substitution, *Allen-Uzawa* elasticity of substitution, *Morishima* elasticity of substitution, *McFadden* elasticity of substitution.

1. Introduction

One of most widely known concepts in economic theory is that of Elasticity of Substitution (ES). Even though it is an old concept, the empirical estimation of ES only became popular after the appearance of the CES and the flexible functional form production functions (translog, quadratic, Leontief). Usually, the objective is to show if the factors of production are substitutes or complements in the production process. Empirical estimates of ES were the object of countless works in the 70s and 80s due to the great interest in knowing the nature of substitutability between capital and energy after the famous petroleum crises.

Originally, the ES concept was used to show how relative shares of labor and capital in the total income vary with changes in quantities of the factors. A review and development of the concept showed that it could be used to classify factors as substitutes or complements and to evaluate the degree of substitution between factors of production. It is less frequently used to analyze substitution among products in a transformation curve and among consumption goods in an indifference curve.

Concentrating on the theory of production, this work has the main objective of doing a brief review of the concept of ES, showing its different definitions and applying it. It tries to show that, when there are more than two production factors, there are several definitions of ES. Unless the differences among alternative definitions are visualized and the importance of each one is understood, the relevance or even the existence of the concept can be questioned. Data of electrified rural properties of the State of Minas Gerais were used to illustrate the calculation and the comparison of the different elasticities. The main objective was to evaluate the nature and the substitutability degree between electric energy and other production factors, mainly petroleum products.

2. Characteristics of production technology

Consider the production technology represented by a general production function,

$$Y_i = f\ (X_{i1}, X_{i2}, X_{i3}, \ldots, X_{ik}) \qquad (1)$$

where Y_i is the amount produced per unit of time; $X_{i1}, X_{i2}, X_{i3}, \ldots, X_{ik}$ are the amounts used of the production factors per unit of time; f represents the production function, and $i = 1,2,3,\ldots, N$ are producing units. In order to represent the production technology, function (1) has to possess a series of basic properties that would turn it useful for economic analysis. Concisely, those properties are a real and finite domain, monotonicity, continuity, concavity and twice differentiable (Chambers, 1988; and Fuss et al., 1978). Monotonicity assures positive marginal products and concavity means decreasing marginal rates of substitution.

An important aspect of the production process is that the same product volume can be obtained with different combinations of factors, that is, a factor can be substituted by another without affecting the production level. Measures of the substitution possibilities among the factors are important for decision making. One of those measures is the Marginal Rate of Substitution (MRS). The MRS of factor i by factor j (MRS_{ij}) is defined as the number of units that factor i is decreased when the use of factor j is increased by one unit, keeping the production level and the amount of the other factors constant. To determine how the amount of factor X_i adjusts to changes in the level of factor X_j, keeping the production level and the amount of the other factors constant, it is enough to differentiate (1) in relation to X_j obtaining:

$$dY = \frac{\partial f}{\partial X_i}\frac{\partial X_i}{\partial X_j} + \frac{\partial f}{\partial X_j} = 0$$

By solving the factors ratio, the MRS of factor i by factor j, is given by

$$MRS_{ij} = -\frac{\partial X_i}{\partial X_j} = \frac{\partial f / \partial X_j}{\partial f / \partial X_i} = \frac{f_j}{f_i} \qquad (2)$$

where f_i and f_j are the marginal products of the factors X_i and X_j, respectively.

Given the condition that the isoquant (curve that represents combinations of factors to produce the same amount of product) is convex in relation to the origin, the value of MRS decreases as X_j increases (and X_i decreases) along an isoquant. The condition of convexity of the isoquant is, then, an expression of the principle of decreasing MRS, based on the presumption that it is more and more difficult to substitute a factor as the substitution progresses.

3. Elasticity of Substitution

The marginal rate of substitution depends on the units used to measure the amounts of the production factors. A similar measure, free from scale and of easier interpretation, is the Elasticity of Substitution (ES).

The concept of ES was introduced by Hicks (1932). Initially, that author did not give a very precise definition of the concept. He defined

ES as a measure of the easiness with which a factor can be substituted by another and presented a mathematical formulation of the concept. A more precise definition was presented by Robinson (1933), who stated that ES is the percentage change in the ratio of factors divided by the percentage change in the ratio of their marginal products, that is, the marginal rate of substitution among the two factors. After the introduction of the concept by Hicks and Robinson, a lot of discussions about the meaning and interpretation of the ES appeared in the literature, specially in works by, Hicks (1933), Kahn (1933,1935), Lerner (1933, 1934, 1936), Meade (1934a, 1934b), Machlup (1935), Pigou (1934), Sweezy (1933) and Tarshis (1934). A revision of the different uses of ES can be seen in Morrissett (1953).

The concept of ES was used by Hicks to analyze the relative shares of labor and capital in the total income, resultanting from changes in the relative amounts of the factors. Basically, the objective was to have a measure of the effects of changes in the price of a factor over its participation in the generated total income, that is to say, in the distribution of income. Initially, it was not objective to use ES to classify factors as substitutes and complements and to evaluate the easiness with which a factor can be substituted by another, due to changes in the relative prices. The use of the concept with that objective is attributed to Robinson (1933). The definition of elasticity of substitution is relatively simple when there are two production factors. When there are three or more factors, there are some complications. So, the two cases should be analyzed separately.

a) Case of Two Factors of Production

Consider the production function $Y = f(X_1, X_2)$, where Y is the amount of product, and X_1 and X_2 are the amounts used of the two factors. According to Hicks (1932) and Robinson (1933), the substitution elasticity (σ_{21}) of factor X_2 by factor X_1 (X_1 is increased and X_2 is decreased) is defined as the percentage change in the quantity of the factor ratio divided by the percentage change in the marginal rate of substitution, whereas the production level remains constant. Since the marginal rate of substitution equals the ratio of the marginal products of the factors, we have

$$\sigma_{21} = \frac{\dfrac{d\left(X_2/X_1\right)}{X_2/X_1}}{\dfrac{d\left(f_1/f_2\right)}{f_1/f_2}} = \frac{d[\log(X_2/X_1)]}{d[\log(f_1/f_2)]} \qquad (3)$$

Graphically, ES measures the degree of curvature of the isoquant. In Figure 1, the initial combination of the factors is represented by point A. At this point, the factor ratio is given by the slope of the ray OA, that is, x_2/x_1, while the MRS_A is given by the slope of tangent ST to the isoquant. Suppose the combination of factors changes to point B. Now, the factor ratio is given by the slope of the ray OB, that is, x_2'/x_1', and for MRS_B, it is given by the slope of tangent S'T'. A measure of the degree of curvature of the isoquant, which represents the level of substitution easiness among the factors, is given by the ratio between the change in the factor ratio and the change in the MRS among the factors. The elasticity of substitution is this ratio, in percentage terms, which will make independent from measurement units.

Depending on the technical conditions of production, there may be a great change in the ratio of factors associated to a small change in the MRS (slope of the isoquant), or a small change in the ratio of factors associated to a great change in MRS. If, for a small change in MRS, it is possible to accomplish a great change in the combination of factors, the elasticity of substitution will be high. This means that it is possible to substitute a great amount of X_1 by X_2, without a great change in MRS, that is to say, it is technically " easy " to substitute a factor by another.

High values for the elasticity of substitution are associated to isoquants that have a small curvature, while small values are associated to isoquants that possess a strong curvature. Figure 2 shows three types of isoquants with their respective substitution elasticities values.

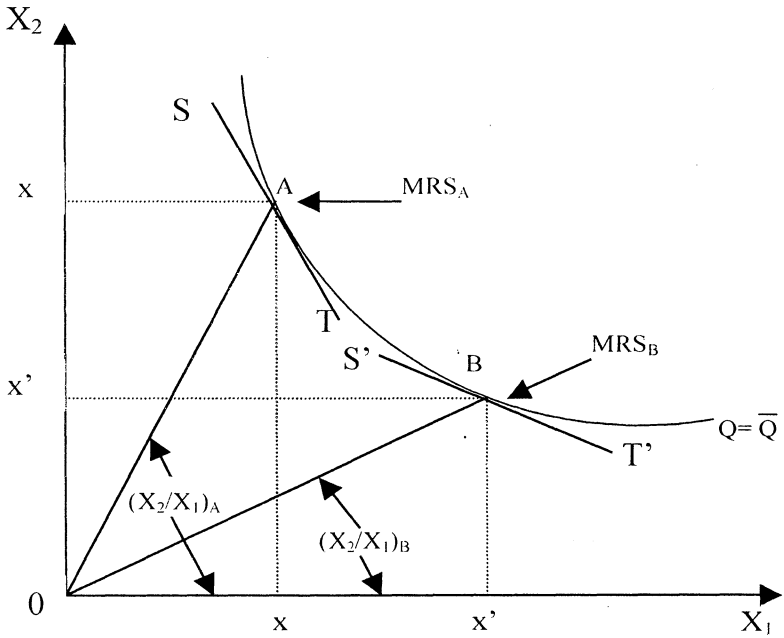

Figure 1 – Graphic Representation of the Elasticity of Substitution.

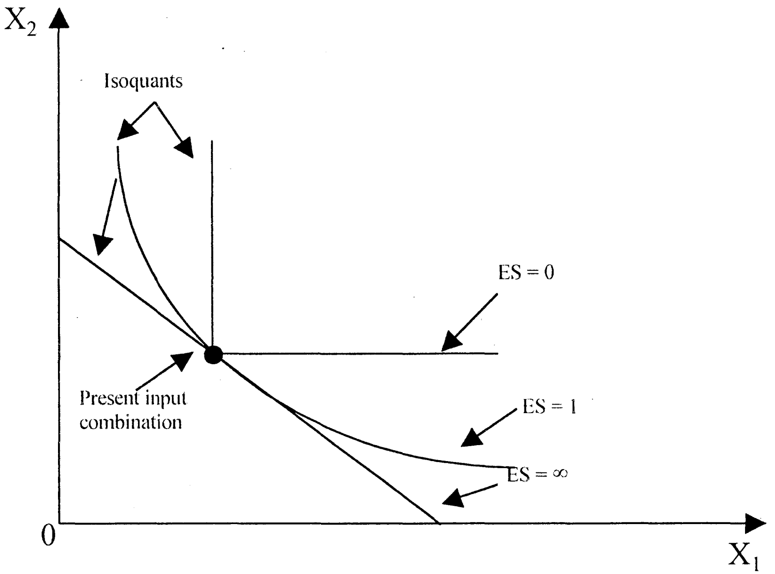

Figure 2 – Relationship between Forms of Isoquants and Values of the Elasticity of Substitution.

Substitution elasticity can also be expressed in terms of factor prices. Given that in competitive equilibrium, the ratio between the marginal products of the factors equals the respective price ratio, equation (3) can be written as

$$\sigma_{21} = \frac{\dfrac{d(X_2/X_1)}{X_2/X_1}}{\dfrac{d(w_1/w_2)}{w_1/w_2}} = \frac{d[\log(X_2/X_1)]}{d[\log(w_1/w_2)]} \qquad (4)$$

where w_1 and w_2 are the prices of factors X_1 and X_2, respectively. This formulation was first attributed to Robinson (1933).

Equation (4) relates changes in the relative prices to changes in the equilibrium combination of factors; that is to say, it relates technical conditions of production to market conditions. Thus, the substitution elasticity will indicate the easiness to which the ratio of factors changes due to changes in the relative prices. A high elasticity means that it is relatively easy to substitute a factor by another when the prices change, but the substitution elasticity is determined by the characteristics of production techniques. Supposing profit maximization by the managers and competitive market conditions, policies that affect the relative prices of the factors will be effective, depending on the characteristics of the technology of production (Robinson, 1933).

Expression (3) is difficult to be calculated. Besides that, from the econometric point of view, a formulation based on the production function that can be estimated with empirical data is more interesting. Equation (3) can be written in terms of partial derivatives of the production function, observing that, along an isoquant, X_2 is function of X_1, and that

$$d(X_2/X_1) = \frac{X_1 dX_2 - X_2 dX_1}{X_1^2} \tag{5}$$

and

$$d(f_1/f_2) = \frac{\partial(f_1/f_2)}{\partial X_1} dX_1 + \frac{\partial(f_1/f_2)}{\partial X_2} dX_2 \tag{6}$$

But, from (2), $dX_2 = -\frac{f_1}{f_2} dX_1$, and equations (5) and (6) can be written as

$$d(X_2/X_1) = -\frac{X_1(f_1/f_2) + X_2}{X_1^2} dX_1 \tag{7}$$

and

$$d(f_1/f_2) = -(f_1/f_2)\frac{\partial(f_1/f_2)}{\partial X_2} - \frac{\partial(f_1/f_2)}{\partial X_1})dX_1 \qquad (8)$$

However,

$$\frac{\partial(f_1/f_2)}{\partial X_1} = \frac{f_2 f_{11} - f_1 f_{12}}{f_2^2} \qquad (9)$$

and

$$\frac{\partial(f_1/f_2)}{\partial X_2} = \frac{f_2 f_{12} - f_1 f_{22}}{f_2^2} \qquad (10)$$

Given that

$$\sigma_{21} = \frac{\dfrac{d(X_2/X_1)}{X_2/X_1}}{\dfrac{d(f_1/f_2)}{f_1/f_2}} = \frac{d(X_2/X_1)\dfrac{f_1}{f_2}}{d(f_1/f_2)\dfrac{X_2}{X_1}}. \qquad (11)$$

and substituting (9) and (10) in (8) and (7) and (8) in (11), we get an expression for the elasticity of substitution based directly on the production function, that is,

$$\sigma_{21} = -\frac{f_1 f_2 (X_1 f_1 + X_2 f_2)}{X_1 X_2 (f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2)} \qquad (12)$$

where $f_i = \dfrac{\partial f}{\partial x_i}$ and $f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$, $i, j = 1, 2$.

In matrix form, (12) can be written as

$$\sigma_{21} = \frac{X_1 f_1 + X_2 f_2}{X_1 X_2} \frac{F_{12}}{F} \tag{13}$$

where F is the determinant of the bordered Hessian matrix of the production function, that is,

$$F = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix}$$

and F_{12} is the cofactor associated to f_{12}. If $f(X)$ is continuous and twice differentiable, Young's theorem assures that $f_{12} = f_{21}$, so $F_{12} = F_{21}$. Thus, $\sigma_{12} = \sigma_2$, and the elasticity of substitution for two factors of production is symmetric, that is, the elasticity of substitution of X_1 by X_2 is the same as the elasticity of substitution of X_2 by X_1. Also, it can be shown that for a quasi concave production function with two factors, σ_{12} will always be positive, indicating that the two factors are always substitutes (Chambers, 1988). These results are only valid for the case of two factors of production.

Values for the elasticity of substitution close to zero indicate few possibilities of substitution, while values significantly greater than zero indicate greater flexibility in the adjustment of factor quantities, when relative prices change.

Through equation (4), we can see that the ES is an elasticity of the ratio between the factor quantities in relation to changes in the price ratio. But, according to Mundlak (1968), not all ES definitions preserve this aspect of the original definition.

Observe that price ratio w_1/w_2 varies when w_1, or w_2, or both vary. This will give different measures of substitution between factors, which can be interpreted as elasticity of substitution. Initially, expression (4) can be written as

$$\sigma_{21} = \frac{d\log(X_2/X_1)}{d\log(w_1/w_2)} = \frac{d\log X_2 - d\log X_1}{d\log w_1 - d\log w_2} = \frac{\hat{x}_2 - \hat{x}_1}{\hat{w}_1 - \hat{w}_2} \qquad (14)$$

where the sign ^ means percentage change. This is the Hicks/Robinson definition for the case of two factors of production: percent change in factor ratio divided by percent change in price ratio. Mundlak (1968) observed that, as the price ratio varies, one can define three different measures of substitutability or ES between factors X_i and X_j. The first is the "two-factor—two-price" ES which is exactly the original definition and can be represented by

$$ES_{2P}^{2F} \equiv \frac{\hat{x}_2 - \hat{x}_1}{\hat{w}_1 - \hat{w}_2} \qquad (15)$$

So, this ES measures the effect of changes in the price ratio over the factor ratio.

The second measure is the *"two-factor-one-price"* ES, represented by

$$ES_{1P}^{2F} \equiv \frac{\hat{x}_2 - \hat{x}_1}{\hat{w}_1} \qquad (16)$$

that measures the effect of changes in the price of one factor over the factor ratio.

Finally, the third measure, the *"one-factor-one-price"* ES, given by

$$ES_{1P}^{1F} \equiv \frac{\hat{x}_2}{\hat{w}_1} \qquad (17)$$

which measures the effect of variation in the price of one factor over the quantity used of the other factor. This measure is akin to the compensated price elasticity of factor demand. It will be shown that the different elasticities of substitution, even trying to measure the same thing, have very distinct characteristics and are classified in the different categories defined by Mundlak (1968).

b) Case of three or more factors

In the case of more than two production factors, the concept of elasticity of substitution can be ambiguous. This is why there are several definitions. The problem is summarized by the fact that, in this case, there are several ways of defining partial derivatives, depending on what is maintained constant.

In the case of two factors, it was shown that the substitution elasticity is always positive and, that, the factors are substitutes. A change in the price ratio induces the firm to use larger amounts of the relatively cheaper factor. However, in the case of three or more factors, all the amounts can be adjusted when any price ratio changes. In that case, the substitution elasticities can be whether positive or negative and the factors can come as "substitutes", when larger amounts of one of them is associated to smaller amounts of the other, or as "complements", when larger amounts of one of them is associated to larger amount of the other. In both cases, the elasticity is called elasticity of substitution.

Now, four definitions of substitution elasticities will be presented for more than two factors, which are, basically, generalizations of the definition for the two-production-factors case.

(i) The Direct Elasticity of Substitution

The direct elasticity of substitution (σ_{ij}^{D}) is identical to definition (3) or (4), except for the fact that, besides the production level, the amounts of all the other factors are maintained constant. That generalization, attributed to Hicks and Allen (1934), is called Hicks Elasticity of Substitution (ESH). Naturally, this is a short run concept, because it does not not take into account the adjustments in the amounts of all the factors due to a change in the prices. In practical terms, it is an identical definition to that of two factors.

(ii) The Allen Partial Elasticity of Substitution

A second definition of elasticity of substitution for more than two factors, is due to Allen (1938) and it is simply a generalization of expression (13). The elasticities, in this case, are called Allen Partial Elasticities of Substitution (ESA), and are defined by

$$\sigma_{ij}^{A} = \frac{X_1 f_1 + X_2 f_2 + \ldots\ldots + X_k f_k}{X_i X_j} \frac{F_{ij}}{F}, \qquad (18)$$

$$\sigma_{ij}^{A} = \frac{\sum_i f_i X_i}{X_i X_j} \frac{F_{ij}}{F}, \qquad i, j = 1,2,3,\ldots,k$$

where F is the determinant of the bordered Hessian matrix of the production function, that is,

$$F = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & \cdots & f_k \\ f_1 & f_{11} & f_{12} & \cdots & \cdots & f_{1k} \\ f_2 & f_{21} & f_{22} & \cdots & \cdots & f_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_k & f_{k1} & f_{k2} & \cdots & \cdots & f_{kk} \end{vmatrix}$$

and F_{ij} is the cofactor related to f_{ij} in the determinant F. According to definition (18), σ_{ij}^{A} are symmetric, that is, $\sigma_{ij}^{A} = \sigma_{ji}^{A}$ for all $i \neq j$. For two factors of production, $\sigma_{ij}^{A} = \sigma_{ij}^{D}$. It is observed that σ_{ii}^{A} can be calculated and it could be called "direct" elasticity of substitution, but it does not possess economic meaning. Those values should be negative, indicating that every production factor is complement to itself and this confirms the concavity of the production function.

Given an estimated production function, the ESAs can be calculated by equation (18). However, depending on the functional form of the production function, that calculation can become quite difficult. This may be one reason why the empiric estimate of elasticities of substitution by means of the production function is not as used as the calculation by means of cost or profit functions. Binswanger (1974) highlights the advantages of using the cost function instead of the production function.

The denominated Partial Elasticities of Substitution by Allen-Uzawa are the same as the Allen ES, but are defined through the cost function instead. They possess much simpler forms, what seems to be one more reason for the preference for the cost function in empirical applications. Uzawa (1962) shows that, for homogeneous production functions, (18) can be calculated as

$$\sigma_{ij}^{A} = \frac{C(Y, w)C_{ij}(Y, w)}{C_i(Y, w)C_j(Y, w)} \tag{19}$$

where C is cost of production, Y is the quantity produced, w is a vector of factor $C_{ij}(Y, w) = \frac{\partial^2 C}{\partial w_i \partial w_j}$, $C_i(Y, w) = \frac{\partial C}{\partial w_i}$, $C_j(Y, w) = \frac{\partial C}{\partial w_j}$.

The partial Allen-Uzawa elasticities of substitution are *dual* to Allen's. For most of the cost functions those elasticities are quite easy to be calculated. Moreover, measures of statistical estimation precision (standard errors) for the elasticities are more difficult to be obtained when equation (18) is used.

The ESA stayed for a long time as an adequate measure of the nature and substitution degree among production factors. In the beginning of the 1960-decade, its characteristics and its economic meaning began to be questioned, mainly after the definition of its dual through the cost function.

Originally, ES is a measure of the percentage change in the ratio of two factors, due to a percentage change in the MRS among them or in the price ratio. However, it can be shown that ESA measures change in the amount of one factor due to change in the price or in the marginal productivity of the other factor. In the terminology of Mundlak (1968), ESA is a "one-factor-one-price" elasticity of substitution. In that sense, it becomes very similar to compensated cross price elasticity of factor demand. In fact, Allen (1938) shows that

$$\varepsilon_{ij} \equiv \frac{\partial X_i}{\partial w_j}\frac{w_j}{X_i} = \alpha_j(\sigma_{ij}^A - \eta) \tag{20}$$

Where ε_{ij} is the cross price elasticity of factor demand of factor X_i; w_j is the price of factor X_j; α_j is the total cost share of factor X_j; σ_{ij}^A is the partial elasticity of substitution between X_i and X_j, and η is the product price elasticity of demand.

If the product price is determined in a competitive market it is constant and elasticity η will be irrelevant in (20), so

$$\varepsilon_{ij} = \alpha_j\sigma_{ij}^A \tag{21}$$

or

$$\sigma_{ij}^A = \frac{1}{\alpha_j}\varepsilon_{ij} \tag{22}$$

that is to say, the Allen partial elasticity of substitution is the same as the cross price elasticity of factor demand weighted by the inverse of the factor share, whose price varies, in the total cost. Thus, ESA is a "one-factor-one-price" type of elasticity. It is observed that ε_{ij} stays asymmetric. Allen (1938) uses (22), to classify the factors X_i and X_j as complements ($\sigma_{ij}^A<0$) or substitutes ($\sigma_{ij}^A>0$), when the price of one of the factors varies, whereas the price of the other factors remain constant.

Based in (21) and (22), Chambers (1988) argues that the ESAs do not possess any information more than the compensated cross price elasticity of factor demand. Also, Blackorby and Russell (1989) present several critics to ESA showing that: a) It does not have any quantitative nor qualitative meaning besides the cross price elasticity; b) It does not possess the properties established in the original definition of Hicks; c) It is not an adequate measure of the curvature of the isoquant and, for this reason, it does not measure the "substitution easiness" among factors; d) It does not give information about relative factor shares in income, resulting from changes in the amounts of the factors; e) It cannot be interpreted as the logarithmic derivative of the factor ratio, in relation to MRS or to the price ratio, as it is established in the original definition; f) It does not add anything more to the cross price elasticity to classify factors in substitutes and complements; g) Its symmetry is not a desirable property. Those authors show that asymmetry is an own characteristic of the concept of ES, in the n-dimensional case.

Besides, Thompson and Taylor (1995) argue that ESA is not adequate for the analysis of factor substitution, in the cases where the share of a factor in the total cost is small, because small variations in the use of the factor cause great variations in the ESA.

(iii) The Morishima Elasticity of Substitution

A third generalization of the concept of elasticity of substitution, for more than two factors, is attributed to Morishima (1967) and elaborated by several authors, such as Kuga and Murota (1972), Koizumi (1976), Blackorby and Russell (1981, 1989).

The Morishima Elasticity of Substitution (ESM), between factors X_i and X_j, is defined as the percentage change in the factor ratio divided by the percentage change in the MRS among X_i and X_j, staying constant the production level and all the other marginal rates of substitution, that is,

$$\sigma_{ij}^{M} = \left. \frac{d[\log(X_i / X_j)]}{d[\log(f_j / f_i)]} \right|_{Y,\; f_k / f_i,\; k \neq i,j \;\; e \;\; i \neq j} \tag{23}$$

The ESM is not symmetric and, as shown by Kuga and Murota (1972) σ_{ij}^{M} can be calculated as

$$\sigma_{ij}^{M} = \frac{f_j}{x_i}\frac{F_{ij}}{F} - \frac{f_j}{x_j}\frac{F_{jj}}{F} \tag{24}$$

and, σ_{ji}^{M}, as

$$\sigma_{ji}^{M} = \frac{f_i}{x_j}\frac{F_{ji}}{F} - \frac{f_i}{x_i}\frac{F_{ii}}{F} \tag{25}$$

Since $\sigma_{ij}^{M} \neq \sigma_{ji}^{M}$, the substitution elasticity of X_i by X_j is different from the substitution elasticity of X_j by X_i.

Observing the expression (13), it can be seen that the ESMs are related to the ESAs. Thus,

$$\sigma_{ij}^{M} = \alpha_j(\sigma_{ij}^{A} - \sigma_{jj}^{A}) \tag{26}$$

and

$$\sigma_{ji}^{M} = \alpha_i(\sigma_{ji}^{A} - \sigma_{ii}^{A}) \qquad (27)$$

Where

$$\alpha_j = \frac{f_j x_j}{\sum_j f_j x_j}.$$

It can be observed that σ_{ii}^{M} is not defined, but σ_{ii}^{A} can be calculated. The ESMs are called *full elasticities of substitution.*

In the case of two factors, ESM = ESA, and EAMs is symmetrical. It can be shown that ESM = ESA and, therefore, symmetrical, for all production function with constant elasticity of substitution, like Cobb-Douglas and CES (Kuga and Murota, 1972).

The ESMs possess some important characteristics, among which asymmetry and relationship with ESA in the classification of factors as complements and substitutes can be pointed out. Regarding asymmetry, Blackorby and Russell (1989) showed that, for more than two factors, the substitution elasticities are naturally asymmetric, once the directions taken by the partial derivatives are, in each case, different. Symmetry of elasticity of substitution is apparently a simple concept, but it is difficult to be justified in economic terms.

Consider the factor demands given by

$$x_i = x_i(w_1, w_2, \ldots, w_i, w_j, \ldots, w_n, Y) \qquad (28)$$

The price elasticity of demand for a factor is defined as

$$\varepsilon_{ij} = \frac{\partial x_i}{\partial w_j}\frac{w_j}{x_i}, \quad i, j = 1,2,\ldots,n. \qquad (29)$$

The elasticity of substitution is a measure of the effect of variation in the factor price ratio, w_i/w_j, over the optimum combination of factors quantities, X_j/X_i. The price ratio can vary with changes in w_i or in w_j or in both, but partial differentiation requires that only one price vary at each time. When w_i varies, all price ratios, w_i/w_j, j = 1,2,....,n, $i \neq j$ vary, and all compensated demands adjust. The same percentage variation in w_i/w_j can be obtained with variation in w_j. In this case, the effect over equilibrium quantities will, in general, be different because the partial derivative taken in the direction of the i^th coordinate is different from that taken in the direction of the j^th coordinate, that is to say, the effect over X_j/X_i depends on which price varies. Thus, the asymmetry characteristic is proper of the elasticity of substitution.

Koizumi (1976) showed that ESMs possess a quite intuitive interpretation in terms of the theory of derived factor demand. Combining (22) with (26) and (27), it can be verified that ESM can be written as:

$$\sigma_{ij}^{M} = \varepsilon_{ij} - \varepsilon_{jj} \qquad (30)$$

and

$$\sigma_{ji}^{M} = \varepsilon_{ji} - \varepsilon_{ii} \qquad (31)$$

The meaning of ESM becomes clearer with (30) and (31). Consider σ_{ij}^{M}, that measures the substitution degree of X_j for X_i, and suppose a decrease of w_j. The effect over the equilibrium combination X_j/X_i is divided in two parts: the effect on X_i, given by ε_{ij} and the effect on X_j, given by ε_{jj} .The net substitution effect is obtained by the deduction of the effect that ε_{jj} has on X_j, by the law of demand.

In the terminology of Mundlak (1968), ESM is a "two-factor-one-price elasticity". The right side of the expression (30) can be written as

$$\varepsilon_{ij} - \varepsilon_{jj} = \frac{\partial \log(X_i / X_j)}{\partial \log(w_j)} \qquad (32)$$

what shows that ESM is an adequate measure of the effect of a change in the price of X_j over the ratio of factor quantities. Thus, ESM has a much more relevant economic meaning than ESA. Two factors, X_i and X_j will be substituted by the ESM, if an increase in the price w_j makes the ratio X_i / X_j increases. This is the net effect shown by (30) or (31). According to (30), the elasticity of substitution of Morishima can, then, be defined as the substitution elasticity of X_i by X_j that measures the percentage change in the amount used of X_i, caused by a 1% percent change in the price of X_j, being deduced the percentage effect on X_j (Koizumi, 1976).

The second characteristic of ESMs refers to the relationship with ESAs in the classification of factors as complements or substitutes. Two factors can be complements with ESA($\sigma_{ij}^{A} <$),but substitutes with ESM ($\sigma_{ij}^{M} > 0$).On the other hand, if two factors are substitutes with ESA ($\sigma_{ij}^{A} > 0$), they will also be substitutes with ESM ($\sigma_{ij}^{M} > 0$) Those relationships can be used to decide about the classification of factors as complements and substitutes (Chambers, 1988).

According to Blackorby and Russell (1989), ESM is more indicated for comparisons of results coming from studies that use samples of different sizes, different measures for certain factors, as capital, for example, and models that are estimated with different number of factors. Those authors show that the ESAs are not invariant with relationship to the input omitted in the model, even if these are separable. Thus, the ESAs estimated in a study with three factors are not comparable to those estimated in a study with four factors. In those cases, they suggest to use the ESMs.

(iv) The McFadden Elasticity of Substitution

Among the definitions of elasticity of substitution, perhaps this is the least known. This definition is attributed to McFadden (1963), and the elasticity is called Shadow Elasticity of Substitution (ESS).

This elasticity was originally defined as the dual of ESH. While ESH is defined by the production function, ESS is defined by the cost function, with constant average cost and the prices of the other factors. For McFadden (1963), the ESS is a long run elasticity, because the amounts of the other factors can be adjusted given the constant prices.

The ESS can be calculated as a weighted average of the Morishima elasticities, σ_{ij}^{M} and σ_{ji}^{M} for each pair of factors. They are defined as

$$\sigma_{ij}^{S} = \frac{\alpha_i}{\alpha_i + \alpha_j}\sigma_{ij}^{M} + \frac{\alpha_j}{\alpha_i + \alpha_j}\sigma_{ji}^{M} \qquad (33)$$

In terms of ESA, we have

$$\sigma_{ij}^{S} = \frac{\alpha_i \alpha_j}{\alpha_i + \alpha_j}(2\sigma_{ij}^{A} - \sigma_{ii}^{A} - \sigma_{jj}^{A}) \qquad (34)$$

Where α_i and α_j are the cost shares of factors X_i and X_j, respectively. The ESS is a "two-factor-two-price" type of elasticity and has the important characteristic of preserving the original meaning of the Hicks/Robinson definition.

In synthesis, these are the alternative definitions of elasticity of substitution found in the literature. The fundamental concept is the same, but different interpretations give alternative operational definitions that result in different estimated values. Knowing the different definitions is important so that the concept can be used appropriately.

4. Empirical Analysis

From the theory of production, the empirical estimation of elasticities of substitution can be made by means of production function, cost function, or profit function. The production function approach is the least used. The reasons should be difficulty in the calculation and deficiency of data. The ESAs calculated from cost functions are the most used elasticities in empirical works.

A translog production function will be used in the calculation of the different substitution elasticities among factors of production in the agriculture of the State of Minas Gerais, Brazil. Consider that production technology is represented by a production function of the general form:

$$Y_i = f\ (X_{i1}, X_{i2}, X_{i3}, X_{i4}, \ldots, X_{il}, X_{i,l+1}, \ldots, X_{ik}, X_{i,k+1}, X_{i,k+2}) \qquad (35)$$

where $i = 1,2,3,\ldots,N$ are farms; Y_i is the value of production of the farm i; X_l is the quantity of labor used, $X_2, \ldots, X_l$ are the quantities of energetic inputs used, $X_{l+1}, \ldots, X_k$ are the quantities of other inputs used, X_{k+1} is the amount of land used for production; X_{k+2} is the amount of capital used; an f represents the production function.

Assuming weak separability between land, capital and the other production factors, function (35) can be written as

$$Y_i = f\ [g(X_{i1}, X_{i2}, X_{i3}, X_{i4}, \ldots, X_{il}, X_{i,l+1}, \ldots, X_{ik}), X_{i,k+1}, X_{i,k+2}] \qquad (36)$$

where "g" is a subfunction of production. With this assumption, the relations between production and factors $X_1, X_2, \ldots, X_l, X_{l+1}, \ldots, X_k$

can be estimated by means of *g*. Thus, the production function to be estimated is given by

$$Y_i = g\ (X_{i1}, X_{i2}, \ldots\ldots, X_{il}, X_{i,l+1}, \ldots\ldots, X_{ik}) \qquad (37)$$

The factors of production considered are labor, two energy inputs, petroleum and electricity, and a non energetic input aggregate[2], formed by expenses with other inputs like fertilizer, pesticides, and materials in general. The variables are defined as: Y = value of production per farm, in US$/year; X_1 = quantity of labor used per farm per year, in man-days; X_2 = quantity of energy used from petroleum products (diesel plus gasoline), measured in KgEP (petroleum kilogram equivalent)/year; X_3 = quantity of electric energy used, measured in KgEP/year; X_4 = value of non energetic inputs, in US$/year.

Equation (37), then, becomes[3]

$$Y_i = g\ (X_{i1}, X_{i2}, X_{i3}, X_{i4}), \quad i = 1,2,3,\ldots\ldots, N \qquad (38)$$

Assuming that the *Translog function (transcendental logarithmic)*, proposed by Christensen, Jorgenson and Lau (1971), can represent the true production function[4], equation (38) is estimated by means of:

$$\ln Y_i = \ln \alpha_0 + \sum_{j=1}^{4} \alpha_j \ln X_{ji} + 1/2 \sum_{j=1}^{4}\sum_{k=1}^{4} \beta_{jk} \ln X_{ji} \ln X_{ki} + \varepsilon_i \qquad (39)$$

[2]The elements of this aggregate are considered non-energetic in the sense of direct energy consumed in the production process.
[3] It is important to remind that the specification of production functions involving monetary values requires the basic assumption that products and production factors are marketed in competitive markets and, so, the prices are constant.
[4]The problem of the functional form can be seen in two ways. The first considers the production function *translog* as originally proposed, that is, as a second order approximation of any production function by means of Taylor series. The second, considers the *translog* function as the true production function. The choice of one or another approach has some implications for estimation and hypotheses tests (Burges, 1975).

where $i = 1,2,3,\ldots,N$ are farms; X_{jk}, $j,k = 1,2,3,4$ are the factors of production; Y is the value of production; and ε_i is a normal, independent and constant-variance random error.

In (39), the equality $\beta_{jk} = \beta_{kj}$, $j,k = 1,2,3,4$ is a condition imposed by Young's theorem, by which the second order cross derivatives are equal. For constant returns to scale the following conditions must be satisfied:

$$\sum_{j=1}^{4} \alpha_j = 1$$

$$\sum_{j=1}^{4} \beta_{jk} = \sum_{k=1}^{4} \beta_{kj} = \sum_{j=1}^{4}\sum_{k=1}^{4} \beta_{jk} = 0 \qquad (40)$$

Moreover, for the technology to be represented by a Cobb-Douglas production function, the condition $\beta_{jk} = 0$, $j,k = 1,2,3,4$ must be observed.

After the parameter estimates of production function (39) are obtained, the different elasticities of substitution will be calculated. First, the ESAs are calculated through equation (18), then the ESMs, with equation (26) and (27), and finally, the ESSs, with equation (34). All those elasticities were calculated for the average of the sample.

The data used are from 1041 electrified farms of the State of Minas Gerais, Brazil, collected in 1991, by means of direct interview with the owners, and with the use of previously tested questionnaire (CEMIG/UFV, 1986).

Equation (39) will be estimated by ordinary least square. The analysis of the behavior of the function will be made with basis on several elements. For monotonicity, marginal products have to be positive. Besides, second order derivatives should be negative, indicating that the marginal products are decreasing. For the production function to satisfy the quasi-concavity property, the bordered Hessian matrix needs to be negative semidefinite. This property defines the so-called stability condition of the production function. The bordered Hessian matrix will be negative semidefinite if the values of its successive minors alternate in sign, being the first negative. Those conditions can be verified at each observation. In the present, as it is cross-section data, they will be checked in the average point of the sample.

The estimated parameters of the *translog* production function, with respective standard deviation and test t, are shown in Table 1. The value of R^2 (adjusted for degrees of freedom) was 0.77, indicating good adjustment of the function to the data. Of the 15 estimated parameters, only four were not statistically significant. Based on the condition $t > 1$ as evidence of statistical significance, only three coefficients are not significant.

Table 1 – Estimated Parameters of the *Translog* Production Function, Minas Gerais, Brazil, 1991

Variable	Estimated Parameters	Standard Error	Value of " t " Test
Constant	4.8533	0.2208	21.9805***
$\ln X_1$	0.1059	0.0536	1.9757**
$\ln X_2$	0.2122	0.0418	5.0766***
$\ln X_3$	0.0949	0.0366	2.5929***
$\ln X_4$	0.1140	0.0519	2.1965**
$(\ln X_1)^2$	0.0739	0.0143	5.1678***
$(\ln X_2)^2$	0.0285	0.0071	4.0141***
$(\ln X_3)^2$	0.0542	0.0067	8.0896***
$(\ln X_4)^2$	0.0261	0.0060	4.3500***
$\ln X_1 \ln X_2$	-0.0332	0.0068	4.8823***
$\ln X_1 \ln X_3$	-0.0040	0.0085	0.4706NS
$\ln X_1 \ln X_4$	0.0004	0.0087	0.0460NS
$\ln X_2 \ln X_3$	-0.0089	0.0062	1.4355NS
$\ln X_2 \ln X_4$	0.0008	0.0023	0.3478NS
$\ln X_3 \ln X_4$	-0.0166	0.0075	2.2133**

$R^2 = 0.7729$ $F = 46.07$*** N = 1041

*** Significant at 1%; ** Significant at 5%; NS = Not Significant.

The Maximum Probability ratio test indicated a rejection of the hypothesis of Cobb-Douglas technology, in favor of translog. In the same way, the hypothesis of constant returns to the scale was rejected. There are evidences of decreasing returns to scale, with an estimated scale elasticity of 0.95.

The interpretation of the estimated coefficients (sign and magnitude) from translog models and from other functions that involve multiplicative logarithmic terms is difficult, because they depend on the units of measurement of the variables. Usually, authors mention that those coefficients do not have economic meaning, what is not correct. To interpret the coefficients, one has to accomplish transformations of the units of measure. On the other hand, the elasticities are invariant with relationship to the units of measurement, reason why are analyzed and interpreted (Hunt and Lynk, 1993; Stern, 1995).

All estimated marginal products, at the average point of the sample, were positive. This indicates that the production function satisfies the monotonicity condition at the considered point. In the same way, the second order derivatives of the production function were all negative. The values of successive principal minors; alternate in sign, as required. Thus, it can be considered that the regularity and stability conditions of the estimated production function are satisfied, considering the average point of the sample.

The results show strong coherence among the estimates of the different substitution elasticities. All the elasticities are positive, indicating that all pairs of factors are substitutes. The magnitudes are also similar, what indicates coherence both in the nature as in the substitution degree among the factors.

Even so, some important differences can be observed. The ESAs are shown in Table 2. Those elasticities are symmetrical and "one-factor-one-price"like, that is to say, they show the percentage change in the amount of one factor resulting from a percentage change in the price of the other factor. It is a measure of how a factor adjusts to changes in the price of the other factor. The elasticity of substitution of petroleum products by electric energy (or vice-versa) is 0.317. An increase of 10% in the price of petroleum causes an increase of 3.17% in the amount of electric energy used. In the same way, due to the symmetry, an increase of 10% in the electricity price causes an increase of 3.17% in the amount of petroleum products used. The other elasticities are all larger than unit, what indicates certain easiness of substitution of factors in the productive process of the electrified rural properties of the State of Minas Gerais. Better substitution possibilities exist among labor and petroleum, due to mechanization. Mechanized equipment uses almost exclusively diesel and has great capacity to substitute labor.

The low substitution elasticity between petroleum and electric energy shows that the system of production of the rural properties depends very much on petroleum and that a lot of operations do not have conditions to use electricity as a source of energy. As an example, soil preparation, cultivation and transportation are operations that demand machine and use almost exclusively petroleum. The operations that use electricity as a source of energy, as for example, irrigation, animal care, grain drying, processing, industrialization and storage, are still not very expressive.

The analysis of the ESMs (Table 3) shows, basically, the same substitution characteristics among the factors. However, the meaning and the asymmetry of ESM supply important information. For being an elasticity type " two-factor-one-price ", ESM shows adjustments in the factor ratio, given changes in the price of one factor. To illustrate, consider the substitution elasticity of petroleum by electric energy, σ^{M}_{PE} = 1,119, and the elasticity of substitution of electric energy by petroleum, σ^{M}_{EP} = 1,375. An increase of 10% in the price of electric energy causes an increase of 11.19%, in the ratio DP/EE. On the other hand, an increase of 10% in the price of petroleum causes an increase of 13.75% in the ratio EE/DP. The elasticity is a little larger when the price of petroleum is adjusted. In that way, policies that seek to stimulate the use of electric energy will be more effective, if directed to the prices of petroleum. An increase in the petroleum price, by means of taxation, for example, will be more effective than a decrease in the price of electricity prices, by means of subsidy or other form.

Rochelle and Ferreira Filho (1999) estimated substitution elasticities among factors used in the cotton crop in the State of Sao Paulo, Brazil, and they found complementarity between labor and machine operations with Allen-Uzawa elasticity. However, for the elasticity of Morishima, the same factors are substitutes, when the price of machine operations varies, and complements, when the price of the labor varies.

The ESSs (Table 4) show the percentage change in the factor ratio, given a percentage change in the price ratio. This is a measured type " two-factor-two-price" and is the one that approaches the most to the original definition of Hicks/Robinson. The value $\sigma^{S}_{PE} = 1{,}243$, for example, represents an estimate of the substitution elasticity among petroleum and electric energy and it indicates that, for a change of 10% in the price ratio, the factor ratio increases by 12.43%.

Table 2 – Allen Partial Elasticities of Substitution (ESA), Minas Gerais, Brazil, 1991

Factors	Labor X_1	Petroleum Products X_2	Electric Energy X_3	Other Inputs X_4
X_1 – Labor	-3.341	2.044	1.351	1.132
X_2 – Petroleum Products		-8.552	0.317	1.682
X_3 – Electric Energy			-7.297	1.543
X_4 – Other Inputs				-2.013

Table 3 – Morishima Elasticities of Substitution (ESM), Minas Gerais, Brazil, 1991

Factors	Labor X_1	Petroleum Products X_2	Electric Energy X_3	Other Inputs X_4
X_1 – Labor	-	1.643	1.271	1.276
X_2 – Petroleum Products	1.573	-	1.119	1.500
X_3 – Electric Energy	1.371	1.375	-	1.443
X_4 – Other Inputs	1.307	1.586	1.299	-

Table 4 – McFadden Elasticities of Substitution (ESS), Minas Gerais, Brazil, 1991

Factors	Labor X_1	Petroleum Products X_2	Electric Energy X_3	Other Inputs X_4
X_1 – Labor	-	1.619	1.304	1.294
X_2 – Petroleum Products		-	1.243	1.563
X_3 – Electric Energy			-	1.337
X_4 – Other Inputs				-

5. Conclusions

The main objective of this paper was to present a revision of different definitions of substitution elasticity. It was shown that the concept developed along the time and that the major difficulties in the use of the concept refer to the case of more than two production factors.

Allen partial elasticity of substitution (ESA), in spite of being the most known and used, presents several deficiencies. It was shown that the ESA does not supply any information besides the price elasticity of the compensated demand of the factor. Morishima elasticity of substitution (ESM) presents several important properties, among them the asymmetry. ESM is more adequate to classify production factors in complements and substitutes. The shadow elasticity of substitution (ESS) is the only definition to beconsistent with the original concept of substitution elasticity, by which ES measures the percentage change in the factors ratio, given a percentage change in the prices ratio.

The estimates of ESA, ESM and ESS, to measure the nature and the substitutability degree between electric energy and other production factors in rural electrified farms of the State of Minas Gerais, Brazil, in 1991, were quite coherent. It was observed that every pair of factors are substitutes and that the possibilities of substitution between labor and petroleum products are larger than those of electricity and petroleum energy. The asymmetry of ESM showed that adjustments in the prices of petroleum cause larger impacts than changes in the prices of electric energy.

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